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Normality and Infinite Product Spaces*

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Recently Noble has shown [1] that if every power X^{ω_α} of a T_1 topological space is normal, then X is compact. In this note we give an elementary proof of this result. We base our proof on a theorem of Stone [3] and a lemma. We also give a short proof of the result based on Stone's theorem and a theorem of Morita [2].

1. AN ELEMENTARY PROOF

We first state Stone's theorem.

THEOREM (Stone [3]). *Let ω_α be an uncountable cardinal and let Z be the integers. Then Z^{ω_α} is not normal. Thus if X^{ω_α} is normal, then X is countably compact.*

This is Theorem 3 and its Corollary in [3]. The proof is short and elementary.

The following lemma is known, but the author is not aware of an adequate reference and so a proof of the lemma is also included.

LEMMA. *Let A be an infinite set and suppose that $\prod_{\alpha\in A} X_\alpha$ is countably compact and $f : \prod_{\alpha\in A} X_\alpha \to R$ is continuous. Then there is a countable set B contained in A such that $f = g \circ \pi_B$ where $g : \prod_{\alpha\in B} X_\alpha \to R$ and $\pi_B : \prod_{\alpha\in A} X_\alpha \to \prod_{\alpha\in B} X_\alpha$ is the projection map.*

The weight of X is the minimum cardinal $w(X)$ such that X has a basis of cardinality $w(X)$. The most precise version of Noble's theorem is the following which we now state and prove assuming the Lemma.

THEOREM (Noble [1]). *Let ω_α be an uncountable cardinal with $\omega_\alpha \geqslant w(X)$. Then if X^{ω_α} is normal, then X is compact.*

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Proof of Noble's Theorem. If ω_α is uncountable with $\omega_\alpha \geqslant w(X)$, then $\omega_\alpha \cdot \omega_\alpha = \omega_\alpha$. Thus if X^{ω_α} is not countably compact, then by Stone's theorem $(X^{\omega_\alpha})^{\omega_\alpha} = X^{\omega_\alpha}$ is not normal. Therefore, if X^{ω_α} is normal, then X^{ω_α} is countably compact as is X^{ω_β} for all $\omega_\beta \leqslant \omega_\alpha$ and X itself. Now suppose that X is not compact. Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets with the finite intersection property with empty intersection. We may assume that card $A \leqslant w(X)$. Let $X_\alpha = X$ for all α in A and let Δ be the diagonal in $\prod_{\alpha\in A} X_\alpha$. Let $K = \prod_{\alpha\in A} F_\alpha \subset \prod_{\alpha\in A} X_\alpha$. Then Δ and K are disjoint closed sets in $\prod_{\alpha\in A} X_\alpha = X^{\operatorname{card} A}$. Thus there is a continuous function $f : \prod_{\alpha\in A} X_\alpha \to [0, 1]$ such that f is zero on Δ and one on K. Let $B \subset A$ be countable and let $g : \prod_{\alpha\in B} X_\alpha \to [0, 1]$ with $f = g \circ \pi_B$ by the Lemma. Since X is countably compact and $\{F_\alpha\}_{\alpha\in B}$ has the finite intersection property and is countable, $\bigcap_{\alpha\in B} F_\alpha \neq \varnothing$. However, this implies that $\pi_B(K) \cap \pi_B(\Delta) \neq \varnothing$ and that $f = g \circ \pi_B$ cannot separate K and Δ, a contradiction. This proves the theorem.

Proof of the Lemma. First we will show that for each $\epsilon > 0$ there is a finite set $B_\epsilon \subset A$ such that if x and y are in $\prod_{\alpha\in A} X_\alpha$ with $x_\alpha = y_\alpha$ for all $\alpha \in B_\epsilon$, then $|f(x) - f(y)| < \epsilon$. Suppose not. Then let a_1 and a_2 be elements of $\prod_{\alpha\in A} X_\alpha$ with $|f(a_1) - f(a_2)| \geqslant \epsilon$. Let F_1 be a finite subset of the index set A such that whenever a_i and b differ only on the coordinates not indexed by F_1, then $|f(a_i) - f(b)| < \epsilon/3$. Then let b_1^1 and b_1^2 be such that b_1^i has the same coordinates as a_i for those coordinates indexed by F_1 and having $\pi_\alpha(b_1^1) = \pi_\alpha(b_1^2)$ for all $\alpha \notin F_1$. Then we have $|f(b_1^1) - f(b_1^2)| \geqslant \epsilon/3$. Let a_1' and a_2' have $\pi_\alpha(a_1') = \pi_\alpha(a_2')$ for all $\alpha \in F_1$ with $|f(a_1') - f(a_2')| \geqslant \epsilon$. Then let F_2 be a finite subset of A disjoint from F_1 such that if a_i' and b have $\pi_\alpha(a_i') = \pi_\alpha(b)$ for all $\alpha \in F_1 \cup F_2$, then $|f(a_i') - f(b)| < \epsilon/3$. Then let b_2^i have the same coordinates as a_i' for those coordinates indexed by F_2 with $\pi_\alpha(b_2^1) = \pi_\alpha(b_2^2)$ for all $\alpha \notin F_2$. Then $|f(b_2^1) - f(b_2^2)| \geqslant \epsilon/3$ also. Continuing inductively, one constructs a sequence of disjoint finite subsets $\{F_i\}_{i=1}^\infty$ of A and sequences $\{b_j^i\}_{i=1}^\infty$ for $i = 1$ and 2 of elements of $\prod_{\alpha\in A} X_\alpha$ with $\pi_\alpha(b_j^1) = \pi_\alpha(b_j^2)$ for all $\alpha \notin F_j$ and with

$$|f(b_j^1) - f(b_j^2)| \geqslant \frac{\epsilon}{3}.$$

Since $\prod_{\alpha\in A} X_\alpha$ is countably compact, $\{b_j^1\}_{i=1}^\infty$ must have a limit point b. But since $\pi_\alpha(b_j^2) = \pi_\alpha(b_j^1)$ for all $\alpha \notin F_j$ and the F_j's are all disjoint, b must also be a limit point of $\{b_j^2\}_{i=1}^\infty$. But this is a contradiction since $|f(b_j^1) - f(b_j^2)| \geqslant \epsilon/3$ for all j.

To finish the proof of the Lemma, let B_n be a finite subset of A with $|f(a) - f(b)| < 1/n$ whenever a and b have the property that $\pi_\alpha(a) = \pi_\alpha(b)$ for all $\alpha \in B_n$. Then let $B = \bigcup_{n=1}^{\infty} B_n$. Then if a and b have the property that $\pi_\alpha(a) = \pi_\alpha(b)$ for all $\alpha \in B$, then $f(a) = f(b)$. Thus B is the countable subset of A which is required in the lemma and g can be defined in the natural way.

2. A Short Proof

Let ω_α be a cardinal. Then X is *ω_α-paracompact* if every open cover $\mathcal{U}$ of X of cardinality $\omega_\beta \leqslant \omega_\alpha$ has a locally finite refinement [2].

THEOREM (Morita [2]). *The space $X \times I^{\omega_\alpha}$ is normal if and only if X is normal and ω_α-paracompact.*

Alternate Proof of Noble's Theorem. By Stone's theorem X must be countably compact. We will show that Morita's theorem implies that X is paracompact. The theorem will then follow since countable compactness and paracompactness together imply compactness. Now to show that X must be paracompact. We may assume that X contains two distinct points $\{0, 1\} \subset X$. Then X^{ω_α} contains $X \times \{0, 1\}^{\omega_\alpha}$ as a closed subspace. Thus $X \times \{0, 1\}^{\omega_\alpha}$ must also be normal. Let C be the Cantor set. Then $X \times C^{\omega_\alpha}$ must be normal since it is homeomorphic to $X \times \{0, 1\}^{\omega_\alpha}$. Let $f : C \to I$ be the Cantor ternary map and let $f^{\omega_\alpha} : C^{\omega_\alpha} \to I^{\omega_\alpha}$ be the product map. Then since f^{ω_α} is proper, $Id \times f^{\omega_\alpha} : X \times C^{\omega_\alpha} \to X \times I^{\omega_\alpha}$ is a closed map. Thus $X \times I^{\omega_\alpha}$ is normal as well. By Morita's theorem, X is ω_α-paracompact. Since $w(X) \leqslant \omega_\alpha$, this implies that X is paracompact and the theorem is proved.

References

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